PROJECTED WRITTEN NOTES FROM THE M408D LECTURE ON THURSDAY, MARCH 28, 2024, ON SEC 9.5: A LINEAR FIRST-ORDER DIFFERENTIAL EQUATION APPLICATION, and SEC. 9.4: THE LOGISTICS MODEL OF POPULATION GROWTH

CLASS #20

THE NOTES BEGIN ON THE NEXT PAGE.

RECALL THAT, LAST CLASS, I PRESENTED THIS IVP

THE "LINEAR FIRST-ORDER DIFFERENTIAL EQUATION INITIAL VALUE PROBLEM PRESENTED IN CLASS"

PROBLEM: SOLVE THE FOLLOWING I.V.P.:

$$\boxed{\frac{dx}{dt} = 4.5 - \frac{x}{300+2t}, \quad x(0)=0, \quad t \geq 0}$$

SOLUTION: CONVERTING this D.E. INTO the STANDARD FORM OF A LINEAR FIRST-ORDER D.E.:

$$\frac{dx}{dt} + \left(\frac{1}{300+2t}\right)x = 4.5, \quad x(0)=0.$$

$$\frac{dx}{dt} + \left(\frac{1}{300+2t}\right)x = \frac{9}{2}.$$

FORM: $\frac{dx}{dt} + P(t)\cdot x = Q(t)$ is a LINEAR FIRST-ORDER DIFFERENTIAL EQUATION with Integrating FACTOR $I(t) = e^{\int P(t)\,dt}$.

$$\int P(t)\,dt = \int \frac{1}{300+2t}\,dt = \frac{1}{2}\int \frac{1}{u}\,du \quad \left(\begin{matrix} u = 300+2t \\ du = 2\,dt \end{matrix}\right)$$

$$= \frac{1}{2}\ln|u| = \frac{1}{2}\ln(300+2t) = \ln\left((300+2t)^{\frac{1}{2}}\right)$$

(Note: since $t \geq 0$, $|300+2t| = (300+2t)$.)

So, $I(t) = e^{\int P(t)dt} = e^{\int \frac{1}{300+2t}dt} = e^{\ln\left((300+2t)^{1/2}\right)} = (300+2t)^{\frac{1}{2}} = I(t)$.

To : $\frac{dx}{dt} + \left(\frac{1}{300+2t}\right)x = \frac{9}{2}$, we multiply

to both sides the integrating factor $(300+2t)^{1/2}$,

to get : $(300+2t)^{1/2}\frac{dx}{dt} + (300+2t)^{1/2}(300+2t)^{-1}x = \frac{9}{2}(300+2t)^{1/2}$

$$(300+2t)^{1/2}\frac{dx}{dt} + (300+2t)^{-1/2}x = \frac{9}{2}(300+2t)^{1/2}$$

$$\therefore \left[(300+2t)^{\frac{1}{2}} \cdot x\right]' = \frac{9}{2}(300+2t)^{1/2}$$

$$\therefore (300+2t)^{1/2} \cdot x = \int\left[(300+2t)^{\frac{1}{2}} \cdot x\right]' dx = \int \frac{9}{2}(300+2t)^{1/2} dt$$

$$= \left(\frac{9}{2}\right)\left(\frac{1}{2}\right)\int u^{1/2} du \quad \left(\text{where } u = (300+2t) \text{ and } du = 2dt\right)$$

$$= \frac{9}{4} \times \frac{2}{3} u^{3/2} + C$$

$$(300+2t)^{1/2} \cdot x = \frac{3}{2}(300+2t)^{3/2} + C$$

(multiplying both sides by $(300+2t)^{-1/2}$):

$$x = \frac{3}{2}(300+2t) + C(300+2t)^{-1/2} = \underline{450 + 3t + C(300+2t)^{-1/2}}$$

when $t=0$, $x=0$ and so, $0 = 450 + 3\times 0 + C(300)^{-1/2}$

$$0 = 450 + \frac{C}{\sqrt{300}} \Rightarrow \frac{C}{10\sqrt{3}} = -450$$

$$\boxed{C = -4500\sqrt{3}}$$

So, $\boxed{x(t) = 450 + 3t - 4500\sqrt{3}(300+2t)^{-1/2}}$

THE IVP's SOLUTION!

A Linear First-Order Differential Equation Application

A 600-gal tank is filled with 300 gal of pure water. A spigot is opened above the tank and a salt solution containing 1.5 lb of salt per gallon of solution begins flowing into the tank at a rate of 3 gal/min. Simultaneously, a drain is opened at the bottom of the tank allowing the solution to leave the tank at a rate of 1 gal/min.

What will be the salt content in the tank at the precise moment that the volume of solution in the tank is equal to the tank's capacity (600 gal) ?

Solution: Let $x(t)$ = the # of lbs of salt in the tank at time t minutes. Note: $x(0) = 0$ lbs of salt.

Let $V(t)$ = the Volume (gallons) of solution in the tank at time t.

$V(0) = 300$ gallons.

$V'(0) = 2$ gallons per min $= 3$ gal/min $- 1$ gal/min

$V(t) = 300 + 2t$ gallons of solution in the tank at time t min.

$x'(t) = \frac{dx}{dt}$ = the rate at which the # of lbs of salt in the tank is changing at time t min.

$\frac{dx}{dt} = (\text{RATE IN} - \text{RATE OUT})$ (lbs of salt / min)

(Salt Concentration) × (Rate of flow)

RATE IN $= \left(1.5 \text{ lbs of salt per gallon}\right)\left(3 \text{ gal/min}\right) = 4.5$ lbs of salt/min

4

RATE_{OUT} = How fast the salt is leaving the tank at time t?

$$\text{RATE}_{\text{OUT}} = \left(\begin{array}{c}\text{Salt concentration}\\ \text{of Solution Leaving}\end{array}\right) \times (\text{Rate of FLOW})$$

$$\text{RATE}_{\text{OUT}} = \underline{?\ ?} \times (1 \text{ gal/min})$$

$$\text{RATE}_{\text{OUT}} = \left(\frac{x(t)}{V(t)} \ \frac{\text{lbs of salt}}{\text{gal}}\right)\left(\text{Rate of flow}\right) \text{gal/min}$$

$$\text{Rate}_{\text{OUT}} = \left(\frac{x(t)}{300+2t} \ \frac{\text{lbs of salt}}{\text{gal}}\right)(1 \text{ gal/min})$$

$$\text{Rate}_{\text{OUT}} = \frac{x(t)}{300+2t} \text{ lbs of salt per min}$$

Since $\frac{dx}{dt} = \text{Rate}_{\text{IN}} - \text{RATE}_{\text{OUT}}$

$$\frac{dx}{dt} = 4.5 - \frac{x(t)}{300+2t} \text{ lbs of salt/min.}$$

and $x(0) = 0$, $t \geq 0$

We need to solve this IVP.

This is the same IVP solved in the handout "Linear First-Order D.E. I.V.P. presented in class."

There, we found the solution to the IVP is $x(t) = 450 + 3t - 4500\sqrt{3}\,(300+2t)^{-\frac{1}{2}}$ lbs of salt at time t.

When is $V(t) = 600$ gallon.

Solve $V(t) = 300 + 2t = 600$

$2t = 300$

$t = 150$ minutes

At $t = 150$ min, the volume of solution in the tank is 600 gal.

$x(150) = 450 + 3(150) - 4500\sqrt{3}\,(300 + 2 \times 150)^{-1/2}$

$x(150) = 582$ lbs of salt

(Rounded from 581.8).

MODELS OF POPULATION GROWTH

The MALTHUSIAN MODEL.

Let k be any non-zero real #.

k is called the "Relative Growth Rate of the Population.

$P(t)$ = The population at time t.

$$\frac{dP}{dt} = kP \quad \text{or} \quad P' = kP$$

$$\frac{P'}{P} = k$$

This is an easy separable D.E.

The General Solution: $P(t) = A\,e^{kt}$

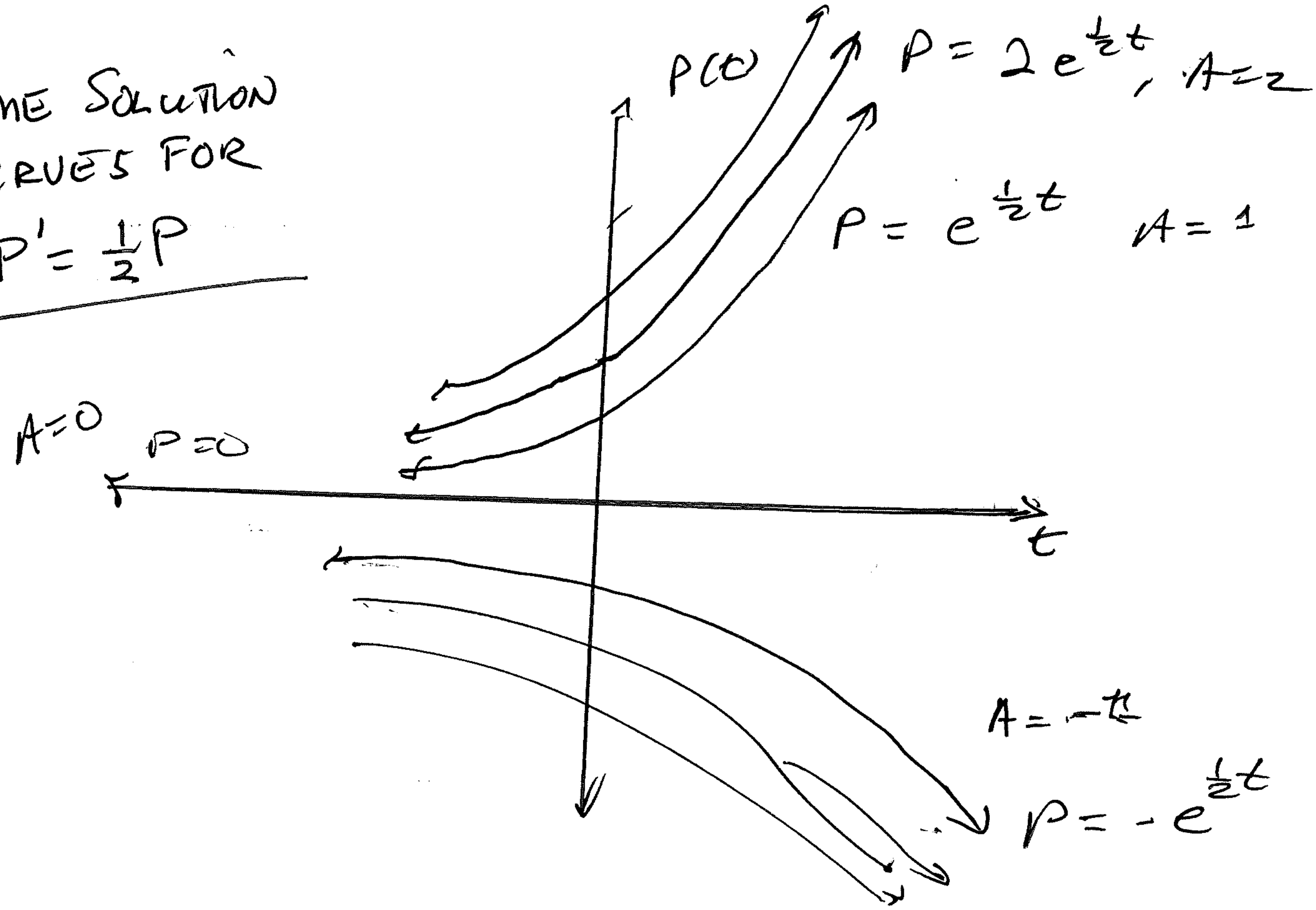
where A is any real number #.

Let's see the solution curves for the solution when $k = \frac{1}{2}$, $P' = \frac{1}{2}P$

with General Solution $P = A\,e^{\frac{1}{2}t}$

SOME SOLUTION CURVES FOR

$P' = \frac{1}{2}P$



The Logistics Model (Sec 9.4)

M = "The Carrying Capacity" = the MAXIMUM Population (Count) that the envirmment can maintain on a sustainable basis.

P = the Population Count at time t.

The Logistics Model

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right), \quad P_0 = \text{The Initial Population}$$

$$P_0 = P(0)$$

The Solution to the IVP of the Logistics Model with Initial condition $P(0) = P_0$ is

$$P = \frac{M}{1 + Ae^{-kt}} \quad \text{where } A = \frac{M - P_0}{P_0}$$

This says that the growth rate is initially close to being proportional to size. In other words, the relative growth rate is almost constant when the population is small. But we also want to reflect the fact that the relative growth rate decreases as the population P increases and becomes negative if P ever exceeds its **carrying capacity** M, the maximum population that the environment is capable of sustaining in the long run. The simplest expression for the relative growth rate that incorporates these assumptions is

$$\frac{1}{P}\frac{dP}{dt} = k\left(1 - \frac{P}{M}\right)$$

Multiplying by P, we obtain the model for population growth known as the **logistic differential equation**:

$$\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)} \qquad \boxed{4}$$

Notice from Equation 4 that if P is small compared with M, then P/M is close to 0 and so $dP/dt \approx kP$. However, if $P \to M$ (the population approaches its carrying capacity), then $P/M \to 1$, so $dP/dt \to 0$. We can deduce information about whether solutions increase or decrease directly from Equation 4. If the population P lies between 0 and M, then the right side of the equation is positive, so $dP/dt > 0$ and the population increases. But if the population exceeds the carrying capacity ($P > M$), then $1 - P/M$ is negative, so $dP/dt < 0$ and the population decreases.

Let's start our more detailed analysis of the logistic differential equation by looking at a direction field.

EXAMPLE 1 Draw a direction field for the logistic equation with $k = 0.08$ and carrying capacity $M = 1000$. What can you deduce about the solutions?

SOLUTION In this case the logistic differential equation is

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)$$

A direction field for this equation is shown in Figure 1. We show only the first quadrant because negative populations aren't meaningful and we are interested only in what happens after $t = 0$.

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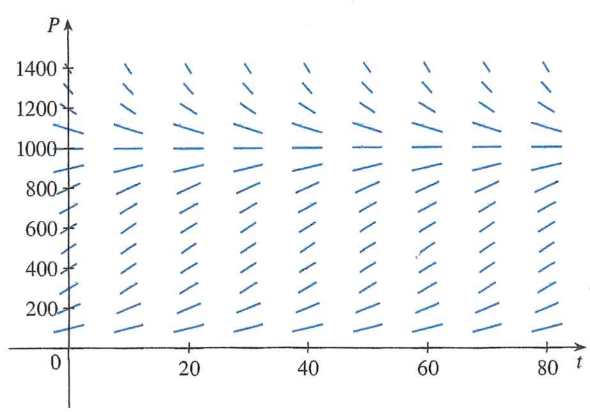

FIGURE 1
Direction field for the logistic equation in Example 1

THE SLOPE FIELD for THE LOGISTICS MODEL $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$.

The logistic equation is autonomous (dP/dt depends only on P, not on t), so the slopes are the same along any horizontal line. As expected, the slopes are positive for $0 < P < 1000$ and negative for $P > 1000$.

The slopes are small when P is close to 0 or 1000 (the carrying capacity). Notice that the solutions move away from the equilibrium solution $P = 0$ and move toward the equilibrium solution $P = 1000$.

In Figure 2 we use the direction field to sketch solution curves with initial populations $P(0) = 100$, $P(0) = 400$, and $P(0) = 1300$. Notice that solution curves that start below $P = 1000$ are increasing and those that start above $P = 1000$ are decreasing. The slopes are greatest when $P \approx 500$ and therefore the solution curves that start below $P = 1000$ have inflection points when $P \approx 500$. In fact we can prove that all solution curves that start below $P = 500$ have an inflection point when P is exactly 500. (See Exercise 13.)

SOME SOLUTIONS ON THE SLOPE FIELD FOR THE LOGISTICS MODEL OF POPULATION GROWTH.

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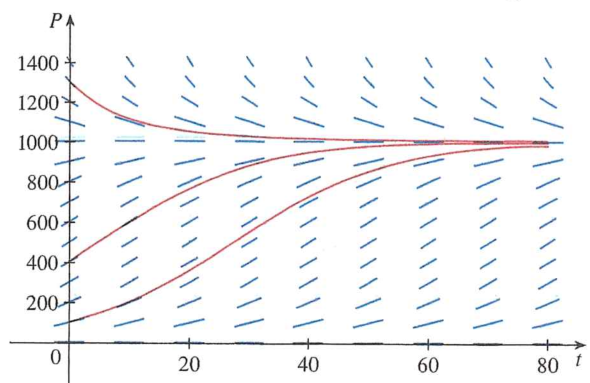

FIGURE 2
Solution curves for the logistic equation in Example 1

The logistic equation (4) is separable and so we can solve it explicitly using the method of Section 9.3. Since

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

we have

$$\boxed{5} \qquad \int \frac{dP}{P(1 - P/M)} = \int k\,dt$$

To evaluate the integral on the left side, we write

$$\frac{1}{P(1 - P/M)} = \frac{M}{P(M - P)}$$

Using partial fractions (see Section 7.4), we get

$$\frac{M}{P(M - P)} = \frac{1}{P} + \frac{1}{M - P}$$

This enables us to rewrite Equation 5:

$$\int \left(\frac{1}{P} + \frac{1}{M - P}\right) dP = \int k\,dt$$

$$\ln |P| - \ln |M - P| = kt + C$$

The Logistic Model --- Statement, Solution, and Application

A population growth function **P** with **Growth Rate k,** with **Carrying Capacity M,** and with **Initial Population** $\mathbf{P(0) = P_0}$, satisfies **The Logistic Model** if it is a solution of the

Logistic Differential Equation: $$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) .$$

The formula for this solution P(t) is:

$$P(t) = \frac{M}{1 + Ae^{-kt}} \quad \text{where} \quad A = \frac{M - P_0}{P_0} .$$

<u>An Application using the solution of the Logistic Differential Equation:</u>

A population obeying the logistic equation begins with 1,000 bacteria, and then it doubles itself in 10 hours.

The population is observed eventually to stabilize at 20,000 bacteria.

(A) Find the number of bacteria in the population after 25 hours and

(B) find the time it takes to reach 1/2 of the carrying capacity.

Solution: Let k = the population's growth rate and M = the carrying capacity.

Let $P(t)$ = the number of bacteria in the population after t hours.

From the information given, M = 20,000 and the initial population is

$P_0 = P(0) = 1{,}000.$ Also, $P(10) = 2{,}000.$

Recall: $P(t) = \dfrac{M}{1 + Ae^{-kt}}$ where $A = \dfrac{M - P_0}{P_0}$.

$P_0 = P(0) = 1{,}000$; $P(10) = 2{,}000$; $M = 20{,}000$

$$A = \frac{20{,}000 - 1{,}000}{1{,}000} = \frac{20-1}{1} = 19 \qquad \boxed{A = 19}$$

$$P = \frac{20{,}000}{1 + 19e^{-kt}}.$$ NEXT, WE FIND k.

$$P(10) = 2{,}000 = \frac{20{,}000}{1 + 19e^{-k(10)}}$$

$$(2{,}000)(1 + 19e^{-k(10)}) = 20{,}000 = 10(2{,}000)$$

$$1 + 19e^{-10k} = 10$$

$$19e^{-10k} = 9$$

$$e^{-10k} = 9/19 \Rightarrow -10k = \ln(9/19)$$

$$10k = -\ln(9/19) = \ln(19/9)$$

$$\boxed{k = \frac{1}{10}\ln(19/9)}$$

$$\boxed{P = \frac{20{,}000}{1 + 19e^{(-\frac{1}{10})\ln(19/9)t}}}$$

(A) For $t = 25$ hours,

$$P(25) = \frac{20{,}000}{1 + 19e^{(-2.5)\ln(19/9)}} = 5083.75$$

After 25 hours, the population is 5,084 bacteria

(B). Find the time it takes to reach ½ of the carrying capacity.

$(\tfrac{1}{2})(20{,}000) = 10{,}000$ and

Recall: $P(t) = \dfrac{20{,}000}{1 + 19e^{(-\frac{1}{10})\ln(19/9)t}}$

Solve $10{,}000 = \dfrac{20{,}000}{1+19e^{(-\frac{1}{10})\ln(19/9)t}}$ for t.

$$(10{,}000)(1 + e^{(-\frac{1}{10})\ln(19/9)t}) = 20{,}000$$

$$1+19e^{(-\frac{1}{10})\ln(19/9)t} = 2$$

$$19e^{(-\frac{1}{10})\ln(19/9)t} = 1$$

$$e^{(-\frac{1}{10})\ln(19/9)t} = 1/19$$ TAKING LOGARITHMS, ...

$$-\tfrac{1}{10}\ln(19/9)t = \ln(1/19)$$

$$\ln(19/9)t = -10\ln(1/19) = 10\ln(19)$$

$$t = \frac{10\ln(19)}{\ln(19/9)} = \frac{10\ln(19)}{\ln(19)-\ln(9)}$$

$t = 39.4055$ (Rounded to 4 decimal places)

It takes 39.4055 hours for the population to reach ½ its carrying capacity.